

* Example 2.

Determine the steady-state temperature distribution in a sphere of radius $r=b$ with boundary surface kept at $T=f(\theta, \phi)$.

This is the same problem as Example 1, except surface temperature is also a function of ϕ . Therefore, it is a 3D problem.

Complete problem, ($\xi \equiv \cos\phi$)

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial T}{\partial \xi} \right] + \frac{1}{r^2(1-\xi^2)} \frac{\partial^2 T}{\partial \phi^2} = 0$$

B.C. in

| | |
|------------------------------|------------------------|
| $T _{r=0}$ finite | ← Natural B.C. |
| $T _{r=b} = f(\theta, \phi)$ | ← Nonhomogeneous ! |
| $T _{\xi=\pm 1}$ finite | ← Natural B.C. |
| $T(\phi+2\pi) = T(\phi)$ | ← periodic requirement |

or, $f(\xi, \phi)$

① Separation of $T(r, \xi, \phi)$.

$$\text{Assume } T(r, \xi, \phi) = R(r) H(\xi) \Phi(\phi)$$

$$\text{then: } R'' H \Phi + \frac{2}{r} R' H \Phi + \frac{R \Phi}{r^2} \frac{d}{d\xi} \left[(1-\xi^2) \frac{dH}{d\xi} \right] + \frac{R H}{r^2(1-\xi^2)} \cdot \Phi'' = 0$$

$$\underbrace{\frac{r^2 R'' + 2r R'}{R}}_{} + \underbrace{\frac{\frac{d}{d\xi} \left[(1-\xi^2) \frac{dH}{d\xi} \right]}{H}}_{} + \underbrace{\frac{\frac{1}{(1-\xi^2)} \Phi''}{\Phi}}_{} = 0$$

$$\frac{r^2 R'' + 2r R'}{R} = - \left\{ \frac{\frac{d}{d\xi} \left[(1-\xi^2) \frac{dH}{d\xi} \right]}{H} + \frac{\frac{1}{(1-\xi^2)} \Phi''}{\Phi} \right\} = \mu_1 \text{ (const.)}$$

$$\text{So. } \underbrace{R^2 R'' + 2rR' - \mu_1 R = 0}$$

$$\text{And: } \frac{\frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta}{d\xi} \right]}{\Theta} + \frac{1}{1-\xi^2} \bar{\Phi}'' = -\mu_1$$

$$\frac{(1-\xi^2) \frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta}{d\xi} \right]}{\Theta} + \frac{\bar{\Phi}''}{\Theta} = -\mu_1 (1-\xi^2) \quad \leftarrow \text{multiplying } (1-\xi^2)$$

$$\text{i.e. } \frac{(1-\xi^2) \frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta}{d\xi} \right]}{\Theta} + \mu_1 (1-\xi^2) = -\frac{\bar{\Phi}''}{\Theta} = \mu_2 \text{ (Const.)}$$

$$\text{So. } \underbrace{\bar{\Phi}'' + \mu_2 \bar{\Phi}} = 0$$

$$\text{And } (1-\xi^2) \frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta}{d\xi} \right] + \left[\mu_1 (1-\xi^2) - \mu_2 \right] \Theta = 0$$

$$\frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta}{d\xi} \right] + \left[\mu_1 - \frac{\mu_2}{1-\xi^2} \right] \Theta = 0$$

$$\text{or, } \underbrace{(1-\xi^2) \Theta'' - 2\xi \Theta' + \left[\mu_1 - \frac{\mu_2}{1-\xi^2} \right] \Theta = 0}$$

② Solving ODEs.

$$\boxed{\begin{cases} R^2 R'' + 2rR' - \mu_1 R = 0 \\ (1-\xi^2) \Theta'' - 2\xi \Theta' + \left[\mu_1 - \frac{\mu_2}{1-\xi^2} \right] \Theta = 0 \\ \bar{\Phi}'' + \mu_2 \bar{\Phi} = 0 \end{cases}}$$

$$\text{Homogeneous B.C. } \left\{ \begin{array}{l} R|_{r=0} = \text{finite} \\ \Theta|_{\xi=\pm 1} = \text{finite} \\ \bar{\Phi}(\phi) = \bar{\Phi}(\phi + 2\pi) \end{array} \right.$$

First, look at equation for $\Phi(\phi)$:

$$\Phi'' + \mu_2 \Phi = 0$$

Imposing B.C. $\Phi(\phi) = \Phi(\phi + 2\pi)$ — periodic B.C.

The only possibility to satisfy the periodic B.C. is to have

$$\boxed{\mu_2 = m^2} \quad m = 0, 1, 2, \dots$$

and:

$$\boxed{\underline{\Phi}_m(\phi) = A \cos m\phi + B \sin m\phi}$$

Second, Look at equation for $\Theta(\xi)$:

$$(1-\xi^2)\Theta'' - 2\xi\Theta' + \left[\mu_1 - \frac{m^2}{1-\xi^2}\right]\Theta = 0$$

Imposing B.C. $\underline{\Theta}|_{\xi=\pm 1} = \text{finite}$ — natural B.C.

The only possibility to satisfy the natural B.C. is to have

$$\boxed{\mu_1 = n(n+1)} \quad n = 0, 1, 2, \dots \text{ and } m \leq n.$$

and

$$\boxed{\underline{\Theta}_{n,m}(\xi) = C P_n^m(\xi) + D Q_n^m(\xi)}$$

P_n^m, Q_n^m are associated Legendre functions of order m)

Because $Q_n^m(\xi)$ diverges at $\xi = \pm 1 \Rightarrow D = 0$

so:

$$\boxed{\underline{\Theta}_{n,m}(\xi) = C P_n^m(\xi)}$$

With the equation written as,

$$(1-\xi^2)\Theta'' - 2\xi\Theta' + \left[n(n+1) - \frac{m^2}{1-\xi^2}\right]\Theta = 0$$

Third, look at equation for $R(r)$.

$$r^2 R'' + 2rR' - n(n+1)R = 0$$

general solution: $\underline{R_n(r) = Er^n + Fr^{-(n+1)}} \quad (n=0, 1, 2, \dots)$

Imposing B.C. $R|_{r=0} = \text{finite}$ — Natural B.C.

Because $r^{-(n+1)} \Big|_{r=0} \rightarrow 0 \Rightarrow F=0$

So: $\underline{\underline{R_n(r) = Er^n}}$

③ Making final solution.

For each $m=0, 1, 2, \dots$ and $n=0, 1, 2, \dots$,

$$\underline{T_{mn}(r, \xi, \phi) = r^n P_n^m(\xi) [A_{mn} \cos m\phi + B_{mn} \sin m\phi]}$$

Therefore:

$$\boxed{T(r, \xi, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r^n P_n^m(\xi) (A_{mn} \cos m\phi + B_{mn} \sin m\phi)} \quad \begin{cases} -1 \leq \xi \leq 1 \\ 0 \leq \phi \leq 2\pi \end{cases}$$

④ Determining unknown coefficient.

Apply nonhomogeneous B.C. $T|_{r=b} = f(\xi, \phi) \quad (\xi \equiv \cos \phi)$

so: $f(\xi, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_n^m(\xi) (A_{mn} \cos m\phi + B_{mn} \sin m\phi) b^n$

Using the orthogonal property for $P_n^m(\xi)$:

$$\int_{-1}^{+1} P_n^m(\xi) P_{n'}^m(\xi) d\xi = \begin{cases} 0 & (n' \neq n) \\ N_n & (n' = n) \end{cases}$$

With:

$$N_n^{(m)} = \int_{-1}^{+1} [P_n^m(\xi)]^2 d\xi = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$$

$$\text{So: } A_{mn} \cos m\phi + B_{mn} \sin m\phi = \frac{1}{\pi N_n^{(m)}} \int_{\phi'=0}^{2\pi} \int_{\xi'=-1}^{+1} \frac{f(\xi', \phi')}{b^n} P_n^m(\xi') \cos[m(\phi - \phi')] d\xi' d\phi'$$

where $\frac{1}{\pi}$ is replaced by $\frac{1}{2\pi}$ for $m=0$.

And the solution becomes:

$$T(r, \xi, \phi) = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \left(\frac{r}{b}\right)^n P_n^m(\xi) \underbrace{\int_{\phi'=0}^{2\pi} \int_{\xi'=-1}^{+1} P_n^m(\xi') \cos[m(\phi - \phi')] f(\xi', \phi') d\xi' d\phi'}_{\sim}$$

where $\frac{1}{\pi}$ is replaced by $\frac{1}{2\pi}$ for $m=0$.